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Renormalization group functions of QCD in the minimal MOM scheme

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Abstract. We provide the full set of renormalization group functions for the renormalization of QCD in the minimal MOM scheme to four loops for the colour group $SU(N_c)$.

1 Introduction.

The main renormalization scheme used in quantum field theory is the modified minimal subtraction, $\overline{\text{MS}}$, scheme introduced in [1, 2]. It has many elegant features which can be exploited to determine the renormalization group functions to a very high loop order. One of these is that of calculability. Briefly, only the divergences with respect to the regulator are removed, together with a specific finite part, $\ln(4\pi e^{-\gamma})$ where γ is the Euler-Mascheroni constant [2]. Ordinarily for conventional perturbation theory calculations one uses dimensional regularization in $d = 4 - 2\epsilon$ dimensions and ϵ plays the role of the regulator. Given this one need only consider the underlying massless quantum field theory safe in the knowledge that in this renormalization scheme the divergences will be mass independent. Thus, since massless Feynman graphs are significantly easier to compute than massive ones, then one can extract the ultraviolet divergences to high loop order. Moreover, using the scheme in gauge theories with massless fields gauge symmetry is preserved, [1]. Thus $\overline{\text{MS}}$ has been established as the favoured scheme for many years. However, for certain problems it is not necessarily the best choice. For instance, in lattice gauge theory computations it is not practical to implement since, for example, it is expensive for calculating Green's functions involving derivatives. Instead physical schemes such as the modified regularization invariant (RI$'$) scheme have been introduced, [3, 4]. These are asymmetric in that the definition is related to the choice of momentum configuration of 3-point functions. Related types of physical schemes, but not motivated by lattice considerations, are the momentum subtraction schemes of [5] denoted by MOM. These differ from RI$'$ schemes in that the 3-point function momentum configuration is completely symmetric. Hence they do not suffer from infrared issues as the configuration does not have exceptional momenta. Both RI$'$ and MOM schemes differ from $\overline{\text{MS}}$ in that finite pieces are absorbed into the renormalization constants which therefore depend on external momentum scales. As a corollary they are more difficult to calculate in analytically to high loop order. Irrespective of which scheme one chooses to use for an analysis, through the structure of the renormalization group equation it is possible to relate results. Thus within perturbation theory one can compute the conversion functions which allow one to map, for example, the coupling constant defined in one scheme to that in another. The other parameters and renormalization group functions can equally be related by the same formalism.

A more recent development has been the introduction of another variant within the RI$'$ and MOM family of physical renormalization schemes, [6]. It is called the minimal MOM scheme and is motivated by a property of the ghost-gluon vertex of QCD in the Landau gauge. This property is the non-renormalization of the vertex, [7]. However, the scheme is an extension of the concept beyond this specific gauge in a way which preserves a definition of the coupling constant in terms of the ghost and gluon form factors, [6]. This effective running coupling constant has been the subject of intense interest in recent years due to interesting features at medium and low energies which were noted earlier in [8]. For instance, it is believed that there are dimension two deviations from the expected running when compared to pure perturbation theory. The most recent work, [9], appears to reaffirm this property. With the minimal MOM scheme the effective coupling constant is not only simple to define but from a practical point of view does not require full knowledge of the ghost-gluon vertex function as would be necessary in other schemes [6]. Indeed in spite of being a non-exceptional momentum configuration it is numerically harder to extract a clean signal from the lattice for a fully symmetric vertex such as in the MOM context. In developing the minimal MOM scheme, [6], the four loop QCD β-function was determined for $SU(N_c)$ not only in the Landau gauge but also for a particular formulation of a linear covariant gauge. The full set of renormalization group functions as far as they could be calculated were not given. Therefore, it is the purpose of this note to provide

the wave function renormalization group functions to as far a loop order as is possible. For an arbitrary colour group this is to three loops for the wave functions and four loops for the β-function and quark mass anomalous dimension. Though we will also provide the former to four loops for $SU(N_c)$. We will do this in two ways. The first is the direct evaluation of all the three loop Green's functions where the minimal MOM renormalization scheme definition is implemented directly. The second is by construction of the associated conversion functions and use of the renormalization group equations. This will serve as a check on our computations and allow us to deduce four loop information. One of the reasons for the direct renormalization is that it provides a non-trivial independent check on the results of [6]. There the β-function was adduced from known finite parts of Green's functions given in [10]. As a separate exercise we choose to work in a minor variation of the original minimal MOM scheme and that is to renormalize the gauge parameter, α, in the full ethos of the minimal MOM scheme. In [6] the renormalization of the gauge parameter is completely equivalent to that of the $\overline{\text{MS}}$ scheme. Though our results will equate for the Landau gauge. A similar issue for α arises in the RI$'$ case [11]. Finally, given the interest in the behaviour of the effective coupling constant in the Landau gauge and its power law deviation, we will provide the minimal MOM anomalous dimension of the operator thought to be associated with the dimension two correction which is $\frac{1}{2}A^a_\mu A^{a\,\mu}$ where A^a_μ is the gluon. This is in order to allow one to perform a complete renormalization group running analysis in the minimal MOM scheme for such infrared problems.

The paper is organized as follows. We recall the definition and properties of the minimal MOM scheme in section 2 before recording our results in the subsequent section. These include the renormalization group functions and the conversion functions for an arbitrary colour group. For $\overline{\text{MS}}$ renormalization group functions which are only known at four loops for $SU(N_c)$, we provide the corresponding minimal MOM scheme results in section 4 together with those for the dimension two operator anomalous dimension. A conclusion is provided in section 5.

2 Formalism.

We begin by recalling the definition of the minimal MOM scheme, [6]. First, if we denote bare quantities in the QCD Lagrangian by the subscript $_{\text{o}}$, then in our notation the renormalization constants, Z_i, are given by

$$\begin{aligned} A^{a\,\mu}_{\text{o}} &= \sqrt{Z_A}\, A^{a\,\mu} ~~,~~ c^a_{\text{o}} ~=~ \sqrt{Z_c}\, c^a ~~,~~ \psi^{iI}_{\text{o}} ~=~ \sqrt{Z_\psi}\, \psi^{iI} \\ \alpha_{\text{o}} &= Z^{-1}_\alpha Z_A\, \alpha ~~,~~ m_{\text{o}} ~=~ m Z_m ~~,~~ g_{\text{o}} ~=~ \mu^\epsilon Z_g\, g \end{aligned} \tag{2.1}$$

where A^a_μ is the gluon, c^a is the Faddeev-Popov ghost and ψ^i is the quark. The indices have the ranges $1 \leq a \leq N_A$, $1 \leq i \leq N_F$ and $1 \leq I \leq N_f$ where N_F and N_A are the respective dimensions of the fundamental and adjoint representations of the colour group and N_f is the number of massive quarks each of the same mass m. The coupling constant is g and α is the gauge parameter of the linear covariant gauge. The Landau gauge corresponds to $\alpha = 0$. We use the above definition of the renormalization of α to be consistent with [11]. Throughout we use dimensional regularization in $d = 4 - 2\epsilon$ spacetime dimensions and the mass scale μ is introduced to ensure the coupling constant is dimensionless in d-dimensions. With these formal definitions of the renormalization constants they are then determined explicitly by specifying a scheme to absorb the infinities in the various 2 and 3-point functions of the theory. For instance, $\overline{\text{MS}}$ corresponds to removing only the poles in ϵ together with a certain finite part at some subtraction point.

For momentum subtraction schemes, denoted generally by MOM, the scheme is defined such that at the subtraction point the poles in ϵ together with all the finite part are absorbed into the

renormalization constant, [5]. For the QCD Lagrangian this produces several different schemes since there are several vertices which one can use to define the coupling constant renormalization. Choosing one, say, means that the remaining vertex functions are finite and consistent with the Slavnov-Taylor identities. The variation on this approach introduced in [6] is that the 2-point functions are renormalized using the MOM criterion of [5] but the 3-point vertices are treated differently. Specifically, to ease comparison with lattice analyses the completely symmetric subtraction point of [5] is not used. Instead the asymmetric point is used where the external momentum of an external leg is nullified. Moreover, partly motivated by the non-renormalization of the ghost-gluon vertex in the Landau gauge, [7], the coupling constant renormalization is defined by ensuring that this vertex renormalization constant is the *same* as the $\overline{\mathrm{MS}}$ one. One benefit of this, [6], is that to define the scheme one only needs to know the vertex structure in the $\overline{\mathrm{MS}}$ scheme which reduces work for non-perturbative applications. In our notation, (2.1), this corresponds to, [6],

$$Z_g^{\overline{\mathrm{MS}}} \sqrt{Z_A^{\overline{\mathrm{MS}}}} Z_c^{\overline{\mathrm{MS}}} ~=~ Z_g^{\mathrm{mMOM}} \sqrt{Z_A^{\mathrm{mMOM}}} Z_c^{\mathrm{mMOM}} \tag{2.2}$$

where mMOM denotes the minimal MOM scheme. Though, in this formal definition it is important to appreciate that the variables g and α on either side of the equation are in different schemes. We note that throughout our convention is that when a scheme is specified as a label on a quantity then it is a function of the parameters g and α in that scheme. With (2.2) then all the renormalization constants of massless QCD are defined for the minimal MOM scheme, [6]. As noted earlier in [6] the gauge parameter renormalization was treated as an $\overline{\mathrm{MS}}$ one rather than define it as the full MOM renormalization as used in [5]. Therefore, we will follow the approach of [5] here and have a minimal MOM α. From the practical point of view our results in the Landau gauge will be the same and differ only in the α dependent part.

The procedure we have used is to apply the MINCER algorithm, [12], to the massless QCD Lagrangian and compute all the 2-point functions as well as the ghost-gluon vertex at the asymmetric point. The quark mass anomalous dimension will be discussed later. This algorithm evaluates massless three loop 2-point functions to the finite part in dimensional regularization. It has been encoded, [13], in the symbolic manipulation language FORM, [14], which is our main computational tool. The Feynman diagrams are generated by QGRAF, [15], and the output converted into FORM input notation. As all graphs are evaluated to the finite parts then we can extract the explicit renormalization constants in the minimal MOM. We do this first by renormalizing the 2-point functions before defining the coupling constant renormalization via (2.2). Then one proceeds to the next loop order. In using the definition of the coupling constant renormalization we have to relate the parameters between the schemes. For the gauge parameter this is given by

$$\alpha_{\mathrm{mMOM}}(\mu) ~=~ \frac{Z_A^{\overline{\mathrm{MS}}}}{Z_A^{\mathrm{mMOM}}} \, \alpha_{\overline{\mathrm{MS}}}(\mu) \tag{2.3}$$

where we used the fact that we are in a linear covariant gauge which implies $Z_\alpha = 1$. To ensure a finite expression in ϵ emerges the parameters within Z_A^{mMOM} have to be converted to their $\overline{\mathrm{MS}}$ partners. This is achieved order by order in perturbation theory. We have determined these to three loops and, with $a = g^2/(16\pi^2)$, found

$$\begin{aligned} a^{\mathrm{mMOM}} ~&=~ a + \left[9\alpha^2 C_A + 18\alpha C_A + 169 C_A - 80 N_f T_F \right] \frac{a^2}{36} \\ &+ \left[405\alpha^3 C_A^2 - 486\zeta(3)\alpha^2 C_A^2 + 2835\alpha^2 C_A^2 + 3564\zeta(3)\alpha C_A^2 + 2421\alpha C_A^2 \right. \\ &\quad - 1440\alpha C_A N_f T_F - 6318\zeta(3) C_A^2 + 76063 C_A^2 - 10368\zeta(3) C_A N_f T_F \\ &\quad \left. - 50656 C_A N_f T_F + 20736\zeta(3) C_F N_f T_F - 23760 C_F N_f T_F + 6400 N_f^2 T_F^2 \right] \frac{a^3}{1296} \end{aligned}$$

$$
\begin{aligned}
& + \Big[-10692\zeta(3)\alpha^4 C_A^3 + 8505\zeta(5)\alpha^4 C_A^3 + 41067\alpha^4 C_A^3 - 59292\zeta(3)\alpha^3 C_A^3 \\
& - 4860\zeta(5)\alpha^3 C_A^3 + 293301\alpha^3 C_A^3 - 138024\zeta(3)\alpha^2 C_A^3 - 85050\zeta(5)\alpha^2 C_A^3 \\
& + 1315035\alpha^2 C_A^3 + 46656\zeta(3)\alpha^2 C_A^2 N_f T_F - 322056\alpha^2 C_A^2 N_f T_F \\
& + 3355020\zeta(3)\alpha C_A^3 - 860220\zeta(5)\alpha C_A^3 + 1277496\alpha C_A^3 \\
& - 2115072\zeta(3)\alpha C_A^2 N_f T_F - 581760\alpha C_A^2 N_f T_F + 373248\zeta(3)\alpha C_A C_F N_f T_F \\
& - 427680\alpha C_A C_F N_f T_F + 331776\zeta(3)\alpha C_A N_f^2 T_F^2 + 32256\alpha C_A N_f^2 T_F^2 \\
& - 6552900\zeta(3) C_A^3 - 1896615\zeta(5) C_A^3 + 42074947 C_A^3 - 4499712\zeta(3) C_A^2 N_f T_F \\
& + 2488320\zeta(5) C_A^2 N_f T_F - 38975424 C_A^2 N_f T_F + 15303168\zeta(3) C_A C_F N_f T_F \\
& + 3732480\zeta(5) C_A C_F N_f T_F - 23755968 C_A C_F N_f T_F + 3068928\zeta(3) C_A N_f^2 T_F^2 \\
& + 9209280 C_A N_f^2 T_F^2 + 4603392 C_F^2 N_f T_F - 7464960\zeta(5) C_F^2 N_f T_f T_F \\
& + 1482624 C_F^2 N_f T_F - 6469632\zeta(3) C_F N_f^2 T_F^2 \\
& + 8065152 C_F N_f^2 T_F^2 - 512000 N_f^3 T_F^3 \Big] \frac{a^4}{46656} \; + \; O(a^5)
\end{aligned} \tag{2.4}
$$

and

$$
\begin{aligned}
\alpha^{\mathrm{mMOM}} \;=\; & \alpha + \Big[-9\alpha^2 C_A - 18\alpha C_A - 97 C_A + 80 N_f T_F \Big] \frac{\alpha a}{36} \\
& + \Big[18\alpha^4 C_A^2 - 18\alpha^3 C_A^2 + 190\alpha^2 C_A^2 - 320\alpha^2 C_A N_f T_F - 576\zeta(3)\alpha C_A^2 + 463\alpha C_A^2 \\
& - 320\alpha C_A N_f T_F + 864\zeta(3) C_A^2 - 7143 C_A^2 + 2304\zeta(3) C_A N_f T_F + 4248 C_A N_f T_F \\
& - 4608\zeta(3) C_F N_f T_F + 5280 C_F N_f T_F \Big] \frac{\alpha a^2}{288} \\
& + \Big[-486\alpha^6 C_A^3 + 1944\alpha^5 C_A^3 + 4212\zeta(3)\alpha^4 C_A^3 - 5670\zeta(5)\alpha^4 C_A^3 - 18792\alpha^4 C_A^3 \\
& + 12960\alpha^4 C_A^2 N_f T_F + 48276\zeta(3)\alpha^3 C_A^3 - 6480\zeta(5)\alpha^3 C_A^3 - 75951\alpha^3 C_A^3 \\
& - 8640\alpha^3 C_A^2 N_f T_F - 52164\zeta(3)\alpha^2 C_A^3 + 2916\zeta(4)\alpha^2 C_A^3 + 124740\zeta(5)\alpha^2 C_A^3 \\
& + 92505\alpha^2 C_A^3 - 129600\zeta(3)\alpha^2 C_A^2 N_f T_F - 147288\alpha^2 C_A^2 N_f T_F \\
& + 248832\zeta(3)\alpha^2 C_A C_F N_f T_F - 285120\alpha^2 C_A C_F N_f T_F - 38400\alpha^2 C_A N_f^2 T_F^2 \\
& - 1303668\zeta(3)\alpha C_A^3 + 11664\zeta(4)\alpha C_A^3 + 447120\zeta(5)\alpha C_A^3 + 354807\alpha C_A^3 \\
& + 698112\zeta(3)\alpha C_A^2 N_f T_F - 312336\alpha C_A^2 N_f T_F + 248832\zeta(3)\alpha C_A C_F N_f T_F \\
& - 285120\alpha C_A C_F N_f T_F - 221184\zeta(3)\alpha C_A N_f^2 T_F^2 + 55296\alpha C_A N_f^2 T_F^2 \\
& + 2007504\zeta(3) C_A^3 + 8748\zeta(4) C_A^3 + 1138050\zeta(5) C_A^3 - 10221367 C_A^3 \\
& + 1505088\zeta(3) C_A^2 N_f T_F - 279936\zeta(4) C_A^2 N_f T_F - 1658880\zeta(5) C_A^2 N_f T_F \\
& + 9236488 C_A^2 N_f T_F - 5156352\zeta(3) C_A C_F N_f T_F + 373248\zeta(4) C_A C_F N_f T_F \\
& - 2488320\zeta(5) C_A C_F N_f T_F + 9293664 C_A C_F N_f T_F - 884736\zeta(3) C_A N_f^2 T_F^2 \\
& - 1343872 C_A N_f^2 T_F^2 - 3068928\zeta(3) C_F^2 N_f T_F + 4976640\zeta(5) C_F^2 N_f T_F \\
& - 988416 C_F^2 N_f T_F + 2101248\zeta(3) C_F N_f^2 T_F^2 - 2842368 C_F N_f^2 T_F^2 \Big] \frac{\alpha a^3}{31104} \\
& + O(a^4)
\end{aligned} \tag{2.5}
$$

where $\zeta(z)$ is the Riemann zeta function. The group Casimirs are defined by

$$
\mathrm{Tr} \left(T^a T^b \right) \;=\; T_F \delta^{ab} \quad , \quad T^a T^a \;=\; C_F I \quad , \quad f^{acd} f^{bcd} \;=\; C_A \delta^{ab} \tag{2.6}
$$

where T^a are the generators of the colour group whose structure functions are f^{abc}. In (2.4) and (2.5) the variables on the right hand side are in the $\overline{\mathrm{MS}}$ scheme. For the Landau gauge it

is easy to see that then the parameters coincide. We have checked that (2.4) agrees with the alternative definition of the mapping given in [6] based on the actual finite parts of the gluon and ghost 2-point functions after their $\overline{\mathrm{MS}}$ renormalization.

While we will perform a direct evaluation of the renormalization constants in the minimal MOM, there are several checks which will be carried out. One is to exploit properties of the renormalization group equation which allows one to map the anomalous dimensions deduced in each scheme via conversion functions which are denoted by $C_i(a,\alpha)$ where i will be a label corresponding to a field or a parameter. First, we will perform the explicit renormalization in the minimal MOM and deduce the anomalous dimensions directly. Then we will compute the conversion functions and from these construct the anomalous dimensions indirectly. Thus if we define the conversion functions by

$$C_g^{\mathrm{mMOM}}(a,\alpha) ~=~ \frac{Z_g^{\mathrm{mMOM}}}{Z_g^{\overline{\mathrm{MS}}}} ~~~,~~~ C_\phi^{\mathrm{mMOM}}(a,\alpha) ~=~ \frac{Z_\phi^{\mathrm{mMOM}}}{Z_\phi^{\overline{\mathrm{MS}}}} \tag{2.7}$$

where $\phi \in \{A, c, \psi\}$, then the minimal MOM renormalization group functions are given by

$$\begin{aligned} \beta^{\mathrm{mMOM}}(a_{\mathrm{mMOM}},\alpha_{\mathrm{mMOM}}) ~&=~ \left[\beta^{\overline{\mathrm{MS}}}(a_{\overline{\mathrm{MS}}}) \frac{\partial a_{\mathrm{mMOM}}}{\partial a_{\overline{\mathrm{MS}}}} \right. \\ &\left. ~~~+~ \alpha_{\overline{\mathrm{MS}}} \gamma^{\overline{\mathrm{MS}}}_\alpha(a_{\overline{\mathrm{MS}}},\alpha_{\overline{\mathrm{MS}}}) \frac{\partial a_{\mathrm{mMOM}}}{\partial \alpha_{\overline{\mathrm{MS}}}} \right]_{\overline{\mathrm{MS}} \to \mathrm{mMOM}} \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \gamma_\phi^{\mathrm{mMOM}}(a_{\mathrm{mMOM}},\alpha_{\mathrm{mMOM}}) ~=& \left[\gamma_\phi^{\overline{\mathrm{MS}}} \left(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}} \right) \right. \\ & +\, \beta^{\overline{\mathrm{MS}}} \left(a_{\overline{\mathrm{MS}}} \right) \frac{\partial}{\partial a_{\overline{\mathrm{MS}}}} \ln C_\phi^{\mathrm{mMOM}} \left(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}} \right) \\ & \left. +\, \alpha_{\overline{\mathrm{MS}}} \gamma_\alpha^{\overline{\mathrm{MS}}} \left(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}} \right) \frac{\partial}{\partial \alpha_{\overline{\mathrm{MS}}}} \ln C_\phi^{\mathrm{mMOM}} \left(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}} \right) \right]_{\overline{\mathrm{MS}} \to \mathrm{mMOM}} . \end{aligned} \tag{2.9}$$

Here $\overline{\mathrm{MS}} \to \mathrm{mMOM}$ means that after computing the right hand side the expression will be a function of $\overline{\mathrm{MS}}$ variables and these must therefore be converted to minimal MOM ones. The relations are given by inverting (2.4) and (2.5). One benefit of this formalism is that it can be exploited to produce the *four* loop anomalous dimensions and β-function. The reason for this is that the three loop conversion functions give a four loop contribution to the minimal MOM anomalous dimensions and β-function and as the $\overline{\mathrm{MS}}$ versions of these are known, [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26], then the left hand side can be deduced at four loops.

3 Results.

We now formally record our results. The one loop expressions will be the same as the $\overline{\mathrm{MS}}$ ones since that term is scheme independent. This includes the β-function as we are using a mass dependent renormalization scheme and only in mass independent schemes is the two loop term scheme independent. Moreover, the β-function will be gauge dependent for the same reason. Therefore, we have*

$$\beta^{\mathrm{mMOM}}(a,\alpha) ~=~ -~ [11 C_A - 4 N_f T_F] \frac{a^2}{3}$$

*A data file is attached which gives an electronic version of all our expressions.

$$
\begin{aligned}
+ \; & \Big[- 3\alpha^3 C_A^2 + 10\alpha^2 C_A^2 - 8\alpha^2 C_A N_f T_F + 13\alpha C_A^2 - 8\alpha C_A N_f T_F - 136 C_A^2 \\
& + 80 C_A N_f T_F + 48 C_F N_f T_F\Big] \frac{a^3}{12} \\
+ \; & \Big[-165\alpha^4 C_A^3 + 24\alpha^4 C_A^2 N_f T_F + 108\zeta(3)\alpha^3 C_A^3 - 189\alpha^3 C_A^3 \\
& - 144\alpha^3 C_A^2 N_f T_F - 468\zeta(3)\alpha^2 C_A^3 + 2175\alpha^2 C_A^3 + 144\zeta(3)\alpha^2 C_A^2 N_f T_F \\
& - 1656\alpha^2 C_A^2 N_f T_F - 864\alpha^2 C_A C_F N_f T_F - 1188\zeta(3)\alpha C_A^3 + 3291\alpha C_A^3 \\
& - 1776\alpha C_A^2 N_f T_F - 1152\alpha C_A C_F N_f T_F + 5148\zeta(3) C_A^3 - 38620 C_A^3 \\
& + 6576\zeta(3) C_A^2 N_f T_F + 32144 C_A^2 N_f T_F - 16896\zeta(3) C_A C_F N_f T_F \\
& + 20512 C_A C_F N_f T_F - 3072\zeta(3) C_A N_f^2 T_F^2 - 4416 C_A N_f^2 T_F^2 \\
& - 576 C_F^2 N_f T_F + 6144\zeta(3) C_F N_f^2 T_F^2 - 5888 C_F N_f^2 T_F^2\Big] \frac{a^4}{288} \\
+ \; & \Big[864\zeta(3)\alpha^5 C_A^4 - 3780\zeta(5)\alpha^5 C_A^4 - 11745\alpha^5 C_A^4 + 1728\alpha^5 C_A^3 N_f T_F \\
& + 32472\zeta(3)\alpha^4 C_A^4 + 4140\zeta(5)\alpha^4 C_A^4 - 81549\alpha^4 C_A^4 \\
& - 1440\zeta(3)\alpha^4 C_A^3 N_f T_F - 5040\zeta(5)\alpha^4 C_A^3 N_f T_F + 7200\alpha^4 C_A^3 N_f T_F \\
& + 7776\alpha^4 C_A^2 C_F N_f T_F + 47052\zeta(3)\alpha^3 C_A^4 + 19800\zeta(5)\alpha^3 C_A^4 \\
& - 81873\alpha^3 C_A^4 + 18432\zeta(3)\alpha^3 C_A^3 N_f T_F + 1440\zeta(5)\alpha^3 C_A^3 N_f T_F \\
& - 67752\alpha^3 C_A^3 N_f T_F - 7776\alpha^3 C_A^2 C_F N_f T_F - 397368\zeta(3)\alpha^2 C_A^4 \\
& + 152280\zeta(5)\alpha^2 C_A^4 + 1028898\alpha^2 C_A^4 - 36576\zeta(3)\alpha^2 C_A^3 N_f T_F \\
& - 1098936\alpha^2 C_A^3 N_f T_F + 639360\zeta(3)\alpha^2 C_A^2 C_F N_f T_F \\
& - 790272\alpha^2 C_A^2 C_F N_f T_F + 73728\zeta(3)\alpha^2 C_A^2 N_f^2 T_F^2 + 133632\alpha^2 C_A^2 N_f^2 T_F^2 \\
& + 20736\alpha^2 C_A C_F^2 N_f T_F - 221184\zeta(3)\alpha^2 C_A C_F N_f^2 T_F^2 \\
& + 211968\alpha^2 C_A C_F N_f^2 T_F^2 - 2400708\zeta(3)\alpha C_A^4 + 987660\zeta(5)\alpha C_A^4 \\
& + 1719423\alpha C_A^4 + 1655712\zeta(3)\alpha C_A^3 N_f T_F - 254880\zeta(5)\alpha C_A^3 N_f T_F \\
& - 1817880\alpha C_A^3 N_f T_F + 798336\zeta(3)\alpha C_A^2 C_F N_f T_F \\
& - 1030752\alpha C_A^2 C_F N_f T_F - 617472\zeta(3)\alpha C_A^2 N_f^2 T_F^2 + 391488\alpha C_A^2 N_f^2 T_F^2 \\
& + 31104\alpha C_A C_F^2 N_f T_F - 331776\zeta(3)\alpha C_A C_F N_f^2 T_F^2 \\
& + 317952\alpha C_A C_F N_f^2 T_F^2 + 98304\zeta(3)\alpha C_A N_f^3 T_F^3 - 24576\alpha C_A N_f^3 T_F^3 \\
& + 5509416\zeta(3) C_A^4 + 3090780\zeta(5) C_A^4 - 22106704 C_A^4 \\
& - 1217376\zeta(3) C_A^3 N_f T_F - 5178960\zeta(5) C_A^3 N_f T_F + 23501280 C_A^3 N_f T_F \\
& - 7050240\zeta(3) C_A^2 C_F N_f T_F - 6082560\zeta(5) C_A^2 C_F N_f T_F \\
& + 17477280 C_A^2 C_F N_f T_F - 1654272\zeta(3) C_A^2 N_f^2 T_F^2 \\
& + 1474560\zeta(5) C_A^2 N_f^2 T_F^2 - 5719680 C_A^2 N_f^2 T_F^2 \\
& - 7907328\zeta(3) C_A C_F^2 N_f T_F + 12165120\zeta(5) C_A C_F^2 N_f T_F \\
& - 607104 C_A C_F^2 N_f T_F + 4755456\zeta(3) C_A C_F N_f^2 T_F^2 \\
& + 2211840\zeta(5) C_A C_F N_f^2 T_F^2 - 10861056 C_A C_F N_f^2 T_F^2 \\
& + 344064\zeta(3) C_A N_f^3 T_F^3 + 229376 C_A N_f^3 T_F^3 - 476928 C_F^3 N_f T_F \\
& + 3538944\zeta(3) C_F^2 N_f^2 T_F^2 - 4423680\zeta(5) C_F^2 N_f^2 T_F^2 + 267264 C_F^2 N_f^2 T_F^2 \\
& - 884736\zeta(3) C_F N_f^3 T_F^3 + 1327104 C_F N_f^3 T_F^3 - 2433024\zeta(3) \frac{d_{AA}^{(4)}}{N_A}
\end{aligned}
$$

$$+ 92160 \frac{d_{AA}^{(4)}}{N_A} + 5750784\zeta(3) \frac{d_{FA}^{(4)}}{N_A} N_f - 589824 \frac{d_{FA}^{(4)}}{N_A} N_f$$
$$- 1769472\zeta(3) \frac{d_{FF}^{(4)}}{N_A} N_f^2 + 811008 \frac{d_{FF}^{(4)}}{N_A} N_f^2 \Bigg] \frac{a^5}{10368} ~+~ O(a^6) ~. \tag{3.1}$$

As the four loop $\overline{\mbox{MS}}$ β-function was computed for an arbitrary colour group, [23], the general colour group Casimirs appear. In our notation they are defined by

$$d_{FF}^{(4)} ~=~ d_F^{abcd} d_F^{abcd} ~~,~~ d_{FA}^{(4)} ~=~ d_F^{abcd} d_A^{abcd} ~~,~~ d_{AA}^{(4)} ~=~ d_A^{abcd} d_A^{abcd} \tag{3.2}$$

and the totally symmetric rank 4 colour tensor is defined by, [27],

$$d_R^{abcd} ~=~ \frac{1}{6} \mbox{Tr} \left(T^a T^{(b} T^c T^{d)} \right) \tag{3.3}$$

where the group generators are in the R representation.

For the anomalous dimensions only the three loop $\overline{\mbox{MS}}$ expressions are known for an arbitrary colour group, [21]. Thus to the same order the minimal MOM expressions are

$$\begin{aligned}
\gamma_A^{\mbox{mMOM}}(a,\alpha) ~=~& [3\alpha C_A - 13 C_A + 8 N_f T_F] \frac{a}{6} \\
&+ \Big[-6\alpha^3 C_A^2 + 17\alpha^2 C_A^2 - 16\alpha^2 C_A N_f T_F + 17\alpha C_A^2 - 16\alpha C_A N_f T_F - 170 C_A^2 \\
&\quad + 136 C_A N_f T_F + 96 C_F N_f T_F \Big] \frac{a^2}{24} \\
&+ \Big[-165\alpha^4 C_A^3 + 24\alpha^4 C_A^2 N_f T_F + 54\zeta(3)\alpha^3 C_A^3 - 126\alpha^3 C_A^3 - 144\alpha^3 C_A^2 N_f T_F \\
&\quad - 576\zeta(3)\alpha^2 C_A^3 + 1761\alpha^2 C_A^3 + 144\zeta(3)\alpha^2 C_A^2 N_f T_F - 1512\alpha^2 C_A^2 N_f T_F \\
&\quad - 864\alpha^2 C_A C_F N_f T_F - 774\zeta(3)\alpha C_A^3 + 102\alpha C_A^3 - 288\zeta(3)\alpha C_A^2 N_f T_F \\
&\quad - 600\alpha C_A^2 N_f T_F - 1152\alpha C_A C_F N_f T_F + 3456\zeta(3) C_A^3 - 23032 C_A^3 \\
&\quad + 6288\zeta(3) C_A^2 N_f T_F + 21320 C_A^2 N_f T_F - 16896\zeta(3) C_A C_F N_f T_F \\
&\quad + 19648 C_A C_F N_f T_F - 3072\zeta(3) C_A N_f^2 T_F^2 - 2496 C_A N_f^2 T_F^2 \\
&\quad - 576 C_F^2 N_f T_F + 6144\zeta(3) C_F N_f^2 T_F^2 - 5888 C_F N_f^2 T_F^2 \Big] \frac{a^3}{288} ~+~ O(a^4) \\
\gamma_c^{\mbox{mMOM}}(a,\alpha) ~=~& [\alpha - 3] \frac{C_A a}{4} + \Big[3\alpha^2 C_A - 3\alpha C_A - 34 C_A + 8 N_f T_F \Big] \frac{C_A a^2}{16} \\
&+ \Big[54\zeta(3)\alpha^3 C_A^2 - 45\alpha^3 C_A^2 - 36\zeta(3)\alpha^2 C_A^2 + 216\alpha^2 C_A^2 - 48\alpha^2 C_A N_f T_F \\
&\quad + 42\zeta(3)\alpha C_A^2 + 109\alpha C_A^2 + 96\zeta(3)\alpha C_A N_f T_F - 152\alpha C_A N_f T_F \\
&\quad + 564\zeta(3) C_A^2 - 5196 C_A^2 + 96\zeta(3) C_A N_f T_F + 3608 C_A N_f T_F + 288 C_F N_f T_F \\
&\quad - 640 N_f^2 T_F^2 \Big] \frac{C_A a^3}{192} ~+~ O(a^4) \\
\gamma_\psi^{\mbox{mMOM}}(a,\alpha) ~=~& \alpha C_F a + C_F \Big[3\alpha^2 C_A + 6\alpha C_A + 25 C_A - 6 C_F - 8 N_f T_F \Big] \frac{C_F a^2}{4} \\
&+ \Big[18\zeta(3)\alpha^3 C_A^2 + 27\alpha^3 C_A^2 - 24\alpha^3 C_A C_F - 90\zeta(3)\alpha^2 C_A^2 + 123\alpha^2 C_A^2 \\
&\quad - 36\alpha^2 C_A C_F + 48\alpha^2 C_A N_f T_F - 618\zeta(3)\alpha C_A^2 + 395\alpha C_A^2 + 72\alpha C_A C_F \\
&\quad + 192\zeta(3)\alpha C_A N_f T_F - 64\alpha C_A N_f T_F - 1470\zeta(3) C_A^2 + 3843 C_A^2 \\
&\quad + 576\zeta(3) C_A C_F - 1260 C_A C_F + 384\zeta(3) C_A N_f T_F - 1840 C_A N_f T_F \\
&\quad + 72 C_F^2 - 96 C_F N_f T_F + 128 N_f^2 T_F^2 \Big] \frac{C_F a^3}{48} ~+~ O(a^4) ~.
\end{aligned} \tag{3.4}$$

We have checked explicitly that the gauge parameter satisfies

$$\gamma_\alpha^{\mathrm{mMOM}}(a,\alpha) ~=~ -~ \gamma_A^{\mathrm{mMOM}}(a,\alpha) \tag{3.5}$$

which is a check on our calculation.

Having provided the anomalous dimensions we have checked that they are completely reproduced using the conversion function approach. The explicit forms of these functions are

$$\begin{aligned}
C_g(a,\alpha) ~=~& 1 + \left[-9\alpha^2 C_A - 18\alpha C_A - 169 C_A + 80 N_f T_F \right] \frac{a}{72} \\
&+ \left[243\alpha^4 C_A^2 - 648\alpha^3 C_A^2 + 1944\zeta(3)\alpha^2 C_A^2 - 1242\alpha^2 C_A^2 - 4320\alpha^2 C_A N_f T_F \right. \\
&- 14256\zeta(3)\alpha C_A^2 + 8568\alpha C_A^2 - 2880\alpha C_A N_f T_F + 25272\zeta(3) C_A^2 - 218569 C_A^2 \\
&+ 41472\zeta(3) C_A N_f T_F + 121504 C_A N_f T_F - 82944\zeta(3) C_F N_f T_F + 95040 C_F N_f T_F \\
&\left. - 6400 N_f^2 T_F^2 \right] \frac{a^2}{10368} \\
&+ \left[-3645\alpha^6 C_A^3 + 21870\alpha^5 C_A^3 + 33048\zeta(3)\alpha^4 C_A^3 - 68040\zeta(5)\alpha^4 C_A^3 \right. \\
&- 183951\alpha^4 C_A^3 + 97200\alpha^4 C_A^2 N_f T_F + 754272\zeta(3)\alpha^3 C_A^3 + 38880\zeta(5)\alpha^3 C_A^3 \\
&- 1501740\alpha^3 C_A^3 - 155520\alpha^3 C_A^2 N_f T_F + 206064\zeta(3)\alpha^2 C_A^3 + 680400\zeta(5)\alpha^2 C_A^3 \\
&- 710235\alpha^2 C_A^3 - 1026432\zeta(3)\alpha^2 C_A^2 N_f T_F - 1887840\alpha^2 C_A^2 N_f T_F \\
&+ 2239488\zeta(3)\alpha^2 C_A C_F N_f T_F - 2566080\alpha^2 C_A C_F N_f T_F - 172800\alpha^2 C_A N_f^2 T_F^2 \\
&- 20977056\zeta(3)\alpha C_A^3 + 6881760\zeta(5)\alpha C_A^3 + 3407958\alpha C_A^3 \\
&+ 11259648\zeta(3)\alpha C_A^2 N_f T_F - 4231296\alpha C_A^2 N_f T_F + 1492992\zeta(3)\alpha C_A C_F N_f T_F \\
&- 1710720\alpha C_A C_F N_f T_F - 2654208\zeta(3)\alpha C_A N_f^2 T_F^2 + 778752\alpha C_A N_f^2 T_F^2 \\
&+ 39610296\zeta(3) C_A^3 + 15172920\zeta(5) C_A^3 - 206477857 C_A^3 \\
&+ 21036672\zeta(3) C_A^2 N_f T_F - 19906560\zeta(5) C_A^2 N_f T_F + 170325744 C_A^2 N_f T_F \\
&- 80372736\zeta(3) C_A C_F N_f T_F - 29859840\zeta(5) C_A C_F N_f T_F \\
&+ 141862464 C_A C_F N_f T_F - 14598144\zeta(3) C_A N_f^2 T_F^2 - 28289280 C_A N_f^2 T_F^2 \\
&- 36827136\zeta(3) C_F^2 N_f T_F + 59719680\zeta(5) C_F^2 N_f T_F - 11860992 C_F^2 N_f T_F \\
&\left. + 31850496\zeta(3) C_F N_f^2 T_F^2 - 41711616 C_F N_f^2 T_F^2 + 512000 N_f^3 T_F^3 \right] \frac{a^3}{746496} \\
&+ O(a^4)
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
C_A(a,\alpha) ~=~& 1 + \left[9\alpha^2 C_A + 18\alpha C_A + 97 C_A - 80 N_f T_F \right] \frac{a}{36} \\
&+ \left[810\alpha^3 C_A^2 + 2430\alpha^2 C_A^2 + 5184\zeta(3)\alpha C_A^2 + 2817\alpha C_A^2 - 2880\alpha C_A N_f T_F \right. \\
&- 7776\zeta(3) C_A^2 + 83105 C_A^2 - 20736\zeta(3) C_A N_f T_F - 69272 C_A N_f T_F \\
&\left. + 41472\zeta(3) C_F N_f T_F - 47520 C_F N_f T_F + 12800 N_f^2 T_F^2 \right] \frac{a^2}{2592} \\
&+ \left[-12636\zeta(3)\alpha^4 C_A^3 + 17010\zeta(5)\alpha^4 C_A^3 + 64638\alpha^4 C_A^3 - 51516\zeta(3)\alpha^3 C_A^3 \right. \\
&+ 19440\zeta(5)\alpha^3 C_A^3 + 322947\alpha^3 C_A^3 + 203148\zeta(3)\alpha^2 C_A^3 - 8748\zeta(4)\alpha^2 C_A^3 \\
&- 374220\zeta(5)\alpha^2 C_A^3 + 1094553\alpha^2 C_A^3 + 15552\zeta(3)\alpha^2 C_A^2 N_f T_F \\
&- 303912\alpha^2 C_A^2 N_f T_F + 4636764\zeta(3)\alpha C_A^3 - 34992\zeta(4)\alpha C_A^3 \\
&- 1341360\zeta(5)\alpha C_A^3 + 1457685\alpha C_A^3 - 3670272\zeta(3)\alpha C_A^2 N_f T_F \\
&- 890064\alpha C_A^2 N_f T_F + 746496\zeta(3)\alpha C_A C_F N_f T_F - 855360\alpha C_A C_F N_f T_F
\end{aligned}$$

$$
\begin{aligned}
& + 663552\zeta(3)\alpha C_A N_f^2 T_F^2 + 64512\alpha C_A N_f^2 T_F^2 - 7531056\zeta(3) C_A^3 \\
& - 26244\zeta(4) C_A^3 - 3414150\zeta(5) C_A^3 + 44961125 C_A^3 - 7293888\zeta(3) C_A^2 N_f T_F \\
& + 839808\zeta(4) C_A^2 N_f T_F + 4976640\zeta(5) C_A^2 N_f T_F - 49928712 C_A^2 N_f T_F \\
& + 23514624\zeta(3) C_A C_F N_f T_F - 1119744\zeta(4) C_A C_F N_f T_F \\
& + 7464960\zeta(5) C_A C_F N_f T_F - 37099872 C_A C_F N_f T_F + 5971968\zeta(3) C_A N_f^2 T_F^2 \\
& + 13873536 C_A N_f^2 T_F^2 + 9206784\zeta(3) C_F^2 N_f T_F - 14929920\zeta(5) C_F^2 N_f T_F \\
& + 2965248 C_F^2 N_f T_F - 12939264\zeta(3) C_F N_f^2 T_F^2 + 16130304 C_F N_f^2 T_F^2 \\
& - 1024000 N_f^3 T_F^3 \Big] \frac{a^3}{93312} ~+~ O(a^4) \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
C_c(a,\alpha) ~=~ & 1 + C_A a \\
& + \Big[-36\zeta(3)\alpha^2 C_A + 72\alpha^2 C_A + 72\zeta(3)\alpha C_A - 21\alpha C_A \\
& - 180\zeta(3) C_A + 1943 C_A - 760 N_f T_F \Big] \frac{C_A a^2}{192} \\
& + \Big[-11178\zeta(3)\alpha^3 C_A^2 - 4860\zeta(5)\alpha^3 C_A^2 + 29241\alpha^3 C_A^2 - 56862\zeta(3)\alpha^2 C_A^2 \\
& + 1458\zeta(4)\alpha^2 C_A^2 + 34020\zeta(5)\alpha^2 C_A^2 + 102789\alpha^2 C_A^2 + 254826\zeta(3)\alpha C_A^2 \\
& + 5832\zeta(4)\alpha C_A^2 - 63180\zeta(5)\alpha C_A^2 - 3510\alpha C_A^2 - 67392\zeta(3)\alpha C_A N_f T_F \\
& + 42984\alpha C_A N_f T_F - 728082\zeta(3) C_A^2 + 4374\zeta(4) C_A^2 - 63180\zeta(5) C_A^2 \\
& + 4329412 C_A^2 - 100224\zeta(3) C_A N_f T_F - 139968\zeta(4) C_A N_f T_F - 2650192 C_A N_f T_F \\
& + 684288\zeta(3) C_F N_f T_F + 186624\zeta(4) C_F N_f T_F - 1165104 C_F N_f T_F \\
& + 27648\zeta(3) N_f^2 T_F^2 + 330304 N_f^2 T_F^2 \Big] \frac{C_A a^3}{31104} ~+~ O(a^4) \qquad (3.8)
\end{aligned}
$$

and

$$
\begin{aligned}
C_\psi(a,\alpha) ~=~ & 1 - \alpha C_F a \\
& + \Big[-9\alpha^2 C_A + 8\alpha^2 C_F + 24\zeta(3)\alpha C_A - 52\alpha C_A + 24\zeta(3) C_A \\
& - 82 C_A + 5 C_F + 28 N_f T_F \Big] \frac{C_F a^2}{8} \\
& + \Big[1728\zeta(3)\alpha^3 C_A^2 - 11880\alpha^3 C_A^2 - 5184\zeta(3)\alpha^3 C_A C_F + 12312\alpha^3 C_A C_F \\
& + 3456\zeta(3)\alpha^3 C_F^2 - 5184\alpha^3 C_F^2 + 25272\zeta(3)\alpha^2 C_A^2 - 972\zeta(4)\alpha^2 C_A^2 \\
& - 6480\zeta(5)\alpha^2 C_A^2 - 63747\alpha^2 C_A^2 - 31104\zeta(3)\alpha^2 C_A C_F + 59616\alpha^2 C_A C_F \\
& + 181440\zeta(3)\alpha C_A^2 - 1944\alpha\zeta(4) C_A^2 - 12960\zeta(5)\alpha C_A^2 - 358191\alpha C_A^2 \\
& + 57024\zeta(3)\alpha C_A C_F - 103680\zeta(5)\alpha C_A C_F + 85536\alpha C_A C_F \\
& - 41472\zeta(3)\alpha C_A N_f T_F + 124056\alpha C_A N_f T_F - 11016\alpha C_F^2 - 28512\alpha C_F N_f T_F \\
& + 678024\zeta(3) C_A^2 + 22356\zeta(4) C_A^2 - 213840\zeta(5) C_A^2 - 1274056 C_A^2 \\
& - 228096\zeta(3) C_A C_F - 31104\zeta(4) C_A C_F + 103680\zeta(5) C_A C_F + 215352 C_A C_F \\
& - 89856\zeta(3) C_A N_f T_F + 760768 C_A N_f T_F + 31536 C_F^2 - 82944\zeta(3) C_F N_f T_F \\
& + 68256 C_F N_f T_F - 100480 N_f^2 T_F^2 \Big] \frac{C_F a^3}{5184} ~+~ O(a^4) ~. \qquad (3.9)
\end{aligned}
$$

We use the convention that the variables on the right hand side are in the $\overline{\mathrm{MS}}$ scheme.

While [6] provided the renormalization group functions for massless QCD it is possible to deduce the quark mass anomalous dimension to four loops for an arbitrary colour group. This

requires the conversion function for the quark mass renormalization and the four loop $\overline{\rm MS}$ anomalous dimension. The latter has been provided in [24] and [25]. To deduce the former we work in the massless theory but renormalize the associated mass operator by inserting it in a quark 2-point function at zero momentum insertion. This was the procedure used in the original three loop $\overline{\rm MS}$ renormalization of [28, 29]. We then use the renormalization condition that there is no finite part at the subtraction point. In this computational setup we can still use the MINCER algorithm, [12, 13]. Thus we can deduce the renormalization constant and hence the three loop quark mass conversion function which is

$$
\begin{aligned}
C_m(a,\alpha) &= 1 + C_F \left[-\alpha - 4 \right] a \\
& \quad + \left[-18\alpha^2 C_A + 24\alpha^2 C_F - 84\alpha C_A + 96\alpha C_F + 432\zeta(3) C_A - 1285 C_A \right. \\
& \quad \left. - 288\zeta(3) C_F + 57 C_F + 332 N_f T_F \right] \frac{C_F a^2}{24} \\
& \quad + \left[-13122\alpha^3 C_A^2 + 15552\alpha^3 C_A C_F - 7776\alpha^3 C_F^2 + 8748\zeta(3)\alpha^2 C_A^2 - 71685\alpha^2 C_A^2 \right. \\
& \quad - 23328\zeta(3)\alpha^2 C_A C_F + 89424\alpha^2 C_A C_F + 46656\zeta(3)\alpha^2 C_F^2 - 31104\alpha^2 C_F^2 \\
& \quad + 103032\zeta(3)\alpha C_A^2 - 357777\alpha C_A^2 - 334368\zeta(3)\alpha C_A C_F + 573804\alpha C_A C_F \\
& \quad - 31104\zeta(3)\alpha C_A N_f T_F + 113400\alpha C_A N_f T_F - 46656\zeta(3)\alpha C_F^2 - 30132\alpha C_F^2 \\
& \quad + 62208\zeta(3)\alpha C_F N_f T_F - 123120\alpha C_F N_f T_F + 3368844\zeta(3) C_A^2 - 466560\zeta(5) C_A^2 \\
& \quad - 6720046 C_A^2 - 2493504\zeta(3) C_A C_F + 155520\zeta(5) C_A C_F + 2028348 C_A C_F \\
& \quad - 532224\zeta(3) C_A N_f T_F + 186624\zeta(4) C_A N_f T_F + 3052384 C_A N_f T_F \\
& \quad + 451008\zeta(3) C_F^2 + 933120\zeta(5) C_F^2 - 2091096 C_F^2 - 331776\zeta(3) C_F N_f T_F \\
& \quad - 186624\zeta(4) C_F N_f T_F + 958176 C_F N_f T_F - 27648\zeta(3) N_f^2 T_F^2 \\
& \quad \left. - 240448 N_f^2 T_F^2 \right] \frac{C_F a^3}{7776} ~+~ O(a^4) ~. \qquad (3.10)
\end{aligned}
$$

Equipped with this and the result of [24, 25] we find the minimal MOM quark mass anomalous dimension is

$$
\begin{aligned}
\gamma_m^{\rm mMOM}(a,\alpha) &= -~3 C_F a + \left[\alpha^2 C_A - 67 C_A - 6 C_F + 8 N_f T_F \right] \frac{C_F a^2}{4} \\
& \quad + \left[-~3\alpha^3 C_A^2 + 24\alpha^3 C_A C_F - 54\zeta(3)\alpha^2 C_A^2 + 411\alpha^2 C_A^2 + 108\alpha^2 C_A C_F \right. \\
& \quad -~48\alpha^2 C_A N_f T_F + 396\zeta(3)\alpha C_A^2 + 15\alpha C_A^2 + 72\alpha C_A C_F + 48\alpha C_A N_f T_F \\
& \quad +~5634\zeta(3) C_A^2 - 10095 C_A^2 - 4224\zeta(3) C_A C_F + 244 C_A C_F \\
& \quad -~1152\zeta(3) C_A N_f T_F + 3888 C_A N_f T_F - 3096 C_F^2 + 1536\zeta(3) C_F N_f T_F \\
& \quad \left. +~736 C_F N_f T_F - 384 N_f^2 T_F^2 \right] \frac{C_F a^3}{48} \\
& \quad + \left[-1134\alpha^4 C_A^3 C_F \zeta(3) + 2835\alpha^4 C_A^3 C_F \zeta(5) - 10125\alpha^4 C_A^3 C_F \right. \\
& \quad + 8586\alpha^4 C_A^2 C_F^2 + 648\alpha^4 C_A^2 C_F N_f T_F - 2592\alpha^4 C_A C_F^3 \\
& \quad - 35316\zeta(3)\alpha^3 C_A^3 C_F - 1620\zeta(5)\alpha^3 C_A^3 C_F + 45522\alpha^3 C_A^3 C_F \\
& \quad - 15552\zeta(3)\alpha^3 C_A^2 C_F^2 + 43740\alpha^3 C_A^2 C_F^2 - 9072\alpha^3 C_A^2 C_F N_f T_F \\
& \quad + 31104\zeta(3)\alpha^3 C_A C_F^3 - 7776\alpha^3 C_A C_F^3 - 485352\zeta(3)\alpha^2 C_A^3 C_F \\
& \quad - 28350\zeta(5)\alpha^2 C_A^3 C_F + 893025\alpha^2 C_A^3 C_F + 167832\zeta(3)\alpha^2 C_A^2 C_F^2 \\
& \quad + 134784\alpha^2 C_A^2 C_F^2 + 85536\zeta(3)\alpha^2 C_A^2 C_F N_f T_F - 307152\alpha^2 C_A^2 C_F N_f T_F \\
& \quad + 160704\zeta(3)\alpha^2 C_A C_F^3 + 246888\alpha^2 C_A C_F^3 - 82944\zeta(3)\alpha^2 C_A C_F^2 N_f T_F \\
& \quad - 90720\alpha^2 C_A C_F^2 N_f T_F + 31104\alpha^2 C_A C_F N_f^2 T_F^2 - 41472\zeta(3)\alpha^2 C_F^3 N_f T_F
\end{aligned}
$$

$$
\begin{aligned}
& + 799956\zeta(3)\alpha C_A^3 C_F - 286740\zeta(5)\alpha C_A^3 C_F + 295551\alpha C_A^3 C_F \\
& - 417744\zeta(3)\alpha C_A^2 C_F^2 + 223668\alpha C_A^2 C_F^2 - 658368\zeta(3)\alpha C_A^2 C_F N_f T_F \\
& + 115416\alpha C_A^2 C_F N_f T_F - 824256\zeta(3)\alpha C_A C_F^3 + 432864\alpha C_A C_F^3 \\
& + 463104\zeta(3)\alpha C_A C_F^2 N_f T_F - 250560\alpha C_A C_F^2 N_f T_F \\
& + 165888\zeta(3)\alpha C_A C_F N_f^2 T_F^2 - 46080\alpha C_A C_F N_f^2 T_F^2 \\
& + 248832\zeta(3)\alpha C_F^3 N_f T_F + 20736\alpha C_F^3 N_f T_F - 110592\zeta(3)\alpha C_F^2 N_f^2 T_F^2 \\
& + 27648\alpha C_F^2 N_f^2 T_F^2 + 16036470\zeta(3) C_A^3 C_F - 6334605\zeta(5) C_A^3 C_F \\
& - 10139319 C_A^3 C_F - 10029096\zeta(3) C_A^2 C_F^2 + 3421440\zeta(5) C_A^2 C_F^2 \\
& - 2188530 C_A^2 C_F^2 - 15748128\zeta(3) C_A^2 C_F N_f T_F + 2737152\zeta(4) C_A^2 C_F N_f T_F \\
& + 4147200\zeta(5) C_A^2 C_F N_f T_F + 8403640 C_A^2 C_F N_f T_F + 2208384\zeta(3) C_A C_F^3 \\
& + 6842880\zeta(5) C_A C_F^3 - 4669704 C_A C_F^3 + 7091712\zeta(3) C_A C_F^2 N_f T_F \\
& - 2737152\zeta(4) C_A C_F^2 N_f T_F + 1244160\zeta(5) C_A C_F^2 N_f T_F \\
& - 2214048 C_A C_F^2 N_f T_F + 2405376\zeta(3) C_A C_F N_f^2 T_F^2 \\
& - 995328\zeta(4) C_A C_F N_f^2 T_F^2 - 2128192 C_A C_F N_f^2 T_F^2 - 1741824\zeta(3) C_F^4 \\
& - 817128 C_F^4 + 4935168\zeta(3) C_F^3 N_f T_F - 7464960\zeta(5) C_F^3 N_f T_F \\
& + 3509568 C_F^3 N_f T_F - 1327104\zeta(3) C_F^2 N_f^2 T_F^2 + 995328\zeta(4) C_F^2 N_f^2 T_F^2 \\
& - 605568 C_F^2 N_f^2 T_F^2 + 147456\zeta(3) C_F N_f^3 T_F^3 - 2048 C_F N_f^3 T_F^3 \\
& + 1244160\zeta(3) \frac{d_{FA}^{(4)}}{N_F} - 165888 \frac{d_{FA}^{(4)}}{N_F} - 2488320\zeta(3) \frac{d_{FF}^{(4)}}{N_F} N_f \\
& \left. + 331776 \frac{d_{FF}^{(4)}}{N_F} N_f \right] \frac{a^4}{5184} ~+~ O(a^5) \qquad (3.11)
\end{aligned}
$$

which involves the same rank 4 Casimirs as the β-function. We have checked the three loop part by the direct evaluation of the anomalous dimension using the minimal MOM renormalization constant. Given that we are considering a relatively new scheme we have also renormalized the flavour non-singlet vector current. This is important since it is a conserved physical current and its anomalous dimension is zero to all orders in perturbation theory. This is true in all schemes but we have checked this explicitly to three loops by repeating the above quark mass operator renormalization but using the vector current, $\bar{\psi}\gamma^\mu\psi$, instead. With the minimal MOM quark wave function renormalization constants and isolating the Lorentz channel of the Green's function with the inserted current corresponding to the transverse part, we have checked that the vector current renormalization constant is unity to three loops in the minimal MOM scheme. Thus the Slavnov-Taylor identity has been checked to this loop order with the above renormalization.

For more practical purposes it is useful to provide the explicit numerical expressions for $SU(3)$. Thus we have

$$
\begin{aligned}
C_g(a,\alpha) ~=~ & 1 + \left[- 0.375000\alpha^2 - 0.750000\alpha + 0.555556 N_f - 7.041667 \right] a \\
& + \left[0.210937\alpha^4 - 0.562500\alpha^3 - 0.625000\alpha^2 N_f + 0.950346\alpha^2 - 0.416667\alpha N_f \right. \\
& \left. - 7.437954\alpha - 0.1543210 N_f^2 + 24.491186 N_f - 163.359911 \right] a^2 \\
& + \left[- 0.131836\alpha^6 + 0.791016\alpha^5 + 0.585937\alpha^4 N_f - 7.768301\alpha^4 - 0.937500\alpha^3 N_f \right. \\
& - 20.064612\alpha^3 - 0.173611\alpha^2 N_f^2 - 18.480594\alpha^2 N_f + 8.788748\alpha^2 \\
& - 2.423078\alpha N_f^2 + 56.307562\alpha N_f - 530.662942\alpha + 0.085734 N_f^3
\end{aligned}
$$

$$
\begin{aligned}
& - 47.581830 N_f^2 + 1099.935641 N_f - 5176.895449 \Big] a^3 ~+~ O(a^4) \\
C_A(a,\alpha) &= 1 + \Big[0.750000 \alpha^2 + 1.500000 \alpha - 1.111111 N_f + 8.083333 \Big] a \\
& + \Big[2.812500 \alpha^3 + 8.437500 \alpha^2 - 1.666667 \alpha N_f + 31.418274 \alpha + 1.234568 N_f^2 \\
& - 53.912928 N_f + 256.103491] a^2 \\
& + \Big[19.411733 \alpha^4 + 81.359870 \alpha^3 - 13.754707 \alpha^2 N_f + 272.349881 \alpha^2 \\
& + 6.929489 \alpha N_f^2 - 254.788106 \alpha N_f + 1621.114903 \alpha - 1.371742 N_f^3 \\
& + 171.267648 N_f^2 - 2601.166373 N_f + 9357.562431 \Big] a^3 ~+~ O(a^4) \\
C_c(a,\alpha) &= 1 + 3.000000 a + \Big[1.346529 \alpha^2 + 3.072567 \alpha - 5.937500 N_f + 80.935770 \Big] a^2 \\
& + \Big[9.344565 \alpha^3 + 61.885373 \alpha^2 - 5.501305 \alpha N_f + 211.462123 \alpha + 8.765877 N_f^2 \\
& - 431.804136 N_f + 2945.691833] a^3 ~+~ O(a^4) \\
C_\psi(a,\alpha) &= 1 - 1.333333 \alpha a + \Big[-2.722222 \alpha^2 - 11.575317 \alpha + 2.333333 N_f - 25.464206 \Big] a^2 \\
& + \Big[-16.906900 \alpha^3 - 72.363802 \alpha^2 + 23.739312 \alpha N_f - 317.382214 \alpha \\
& - 6.460905 N_f^2 + 246.442650 N_f - 1489.980500 \Big] a^3 ~+~ O(a^4) \\
C_m(a,\alpha) &= 1 + [-1.333333 \alpha - 5.333333] a \\
& + \Big[-1.222222 \alpha^2 - 6.888889 \alpha + 9.222222 N_f - 149.040228 \Big] a^2 \\
& + \Big[-11.953704 \alpha^3 - 44.682370 \alpha^2 + 14.024098 \alpha N_f - 269.395219 \alpha \\
& - 11.731930 N_f^2 + 713.333651 N_f - 5598.952656 \Big] a^3 ~+~ O(a^4) ~.
\end{aligned} \tag{3.12}
$$

Clearly it would appear that the series have large corrections at three loops. Though that for the quark wave function is best.

4 $SU(N_c)$.

Although we have given the minimal MOM scheme results to as high a loop order as is possible for an *arbitrary* colour group, it is possible to provide the complete set at *four* loops for the case of $SU(N_c)$. This is because the four loop $\overline{\mathrm{MS}}$ anomalous dimensions of the gluon, ghost and quark are known for this colour group for an arbitrary linear covariant gauge fixing, [24, 26]. Using the electronically available data files associated with the latter article we have extended the various three loop minimal MOM results using the same method. This is also possible since we have the mapping for the gauge parameter between the two schemes at three loops. Thus for the gluon we have

$$
\begin{aligned}
\gamma_A^{\mathrm{mMOM}}(a,\alpha) &= [3 \alpha N_c - 13 N_c + 4 N_f] \frac{a}{6} \\
& + \Big[-6 \alpha^3 N_c^3 + 17 \alpha^2 N_c^3 - 8 \alpha^2 N_c^2 N_f + 17 \alpha N_c^3 - 8 \alpha N_c^2 N_f - 170 N_c^3 \\
& + 92 N_c^2 N_f - 24 N_f \Big] \frac{a^2}{24 N_c} \\
& + \Big[-165 \alpha^4 N_c^5 + 12 \alpha^4 N_c^4 N_f + 54 \zeta(3) \alpha^3 N_c^5 - 126 \alpha^3 N_c^5 - 72 \alpha^3 N_c^4 N_f \\
& - 576 \zeta(3) \alpha^2 N_c^5 + 1761 \alpha^2 N_c^5 + 72 \zeta(3) \alpha^2 N_c^4 N_f - 972 \alpha^2 N_c^4 N_f \\
& + 216 \alpha^2 N_c^2 N_f - 774 \zeta(3) \alpha N_c^5 + 102 \alpha N_c^5 - 144 \zeta(3) \alpha N_c^4 N_f - 588 \alpha N_c^4 N_f
\end{aligned}
$$

$$
\begin{aligned}
& + 288\alpha N_c^2 N_f + 3456\zeta(3) N_c^5 - 23032 N_c^5 - 1080\zeta(3) N_c^4 N_f + 15500 N_c^4 N_f \\
& - 1360 N_c^3 N_f^2 + 4224\zeta(3) N_c^2 N_f - 4768 N_c^2 N_f - 768\zeta(3) N_c N_f^2 + 736 N_c N_f^2 \\
& - 72 N_f \Big] \frac{a^3}{288 N_c^2} \\
+ \Big[& 1728\zeta(3)\alpha^5 N_c^7 - 7560\zeta(5)\alpha^5 N_c^7 - 23490\alpha^5 N_c^7 + 1728\alpha^5 N_c^6 N_f \\
& + 52389\zeta(3)\alpha^4 N_c^7 - 2250\zeta(5)\alpha^4 N_c^7 - 130590\alpha^4 N_c^7 - 1440\zeta(3)\alpha^4 N_c^6 N_f \\
& - 5040\zeta(5)\alpha^4 N_c^6 N_f + 10440\alpha^4 N_c^6 N_f + 3888\zeta(3)\alpha^4 N_c^5 + 2430\zeta(5)\alpha^4 N_c^5 \\
& - 3888\alpha^4 N_c^4 N_f + 38214\zeta(3)\alpha^3 N_c^7 + 40410\zeta(5)\alpha^3 N_c^7 - 20727\alpha^3 N_c^7 \\
& + 23616\zeta(3)\alpha^3 N_c^6 N_f + 1440\zeta(5)\alpha^3 N_c^6 N_f - 86328\alpha^3 N_c^6 N_f \\
& + 1944\zeta(3)\alpha^3 N_c^5 + 53460\zeta(5)\alpha^3 N_c^5 + 3888\alpha^3 N_c^4 N_f - 937188\zeta(3)\alpha^2 N_c^7 \\
& + 656370\zeta(5)\alpha^2 N_c^7 + 1387395\alpha^2 N_c^7 + 324144\zeta(3)\alpha^2 N_c^6 N_f \\
& - 30240\zeta(5)\alpha^2 N_c^6 N_f - 1222596\alpha^2 N_c^6 N_f - 18432\zeta(3)\alpha^2 N_c^5 N_f^2 \\
& + 93888\alpha^2 N_c^5 N_f^2 + 93312\zeta(3)\alpha^2 N_c^5 + 145800\zeta(5)\alpha^2 N_c^5 - 5832\alpha^2 N_c^5 \\
& - 319680\zeta(3)\alpha^2 N_c^4 N_f + 373104\alpha^2 N_c^4 N_f + 55296\zeta(3)\alpha^2 N_c^3 N_f^2 \\
& - 52992\alpha^2 N_c^3 N_f^2 + 5184\alpha^2 N_c^2 N_f - 3727014\zeta(3)\alpha N_c^7 \\
& + 1557630\zeta(5)\alpha N_c^7 + 1030995\alpha N_c^7 + 1860768\zeta(3)\alpha N_c^6 N_f \\
& - 142560\zeta(5)\alpha N_c^6 N_f - 1254696\alpha N_c^6 N_f - 400896\zeta(3)\alpha N_c^5 N_f^2 \\
& + 192288\alpha N_c^5 N_f^2 + 872856\zeta(3)\alpha N_c^5 - 461700\zeta(5)\alpha N_c^5 - 23328\alpha N_c^5 \\
& + 24576\zeta(3)\alpha N_c^4 N_f^3 - 6144\alpha N_c^4 N_f^3 - 383616\zeta(3)\alpha N_c^4 N_f \\
& + 428544\alpha N_c^4 N_f + 82944\zeta(3)\alpha N_c^3 N_f^2 - 79488\alpha N_c^3 N_f^2 + 7776\alpha N_c^2 N_f \\
& + 8025711\zeta(3) N_c^7 + 4451400\zeta(5) N_c^7 - 27205691 N_c^7 \\
& - 5654160\zeta(3) N_c^6 N_f - 4647600\zeta(5) N_c^6 N_f + 23030340 N_c^6 N_f \\
& + 642816\zeta(3) N_c^5 N_f^2 + 737280\zeta(5) N_c^5 N_f^2 - 4246944 N_c^5 N_f^2 \\
& - 2563488\zeta(3) N_c^5 - 4949910\zeta(5) N_c^5 + 142344 N_c^5 - 24576\zeta(3) N_c^4 N_f^3 \\
& + 188672 N_c^4 N_f^3 + 8424000\zeta(3) N_c^4 N_f - 2730240\zeta(5) N_c^4 N_f \\
& - 7459344 N_c^4 N_f - 1810944\zeta(3) N_c^3 N_f^2 + 552960\zeta(5) N_c^3 N_f^2 \\
& + 2377728 N_c^3 N_f^2 + 110592\zeta(3) N_c^2 N_f^3 - 165888 N_c^2 N_f^3 \\
& - 1976832\zeta(3) N_c^2 N_f + 3041280\zeta(5) N_c^2 N_f - 326736 N_c^2 N_f \\
& - 221184\zeta(3) N_c N_f^2 - 552960\zeta(5) N_c N_f^2 + 337536 N_c N_f^2 \\
& + 59616 N_f \Big] \frac{a^4}{20736 N_c^3} ~+~ O(a^5) \qquad (4.1)
\end{aligned}
$$

where we have substituted the $SU(N_c)$ values for C_F and C_A. Similarly, the ghost anomalous dimension is

$$
\begin{aligned}
\gamma_c^{\mathrm{mMOM}}(a,\alpha) ~=~ & N_c[\alpha - 3]\frac{a}{4} \\
& + N_c \left[3\alpha^2 N_c - 3\alpha N_c - 34 N_c + 4 N_f \right] \frac{a^2}{16} \\
& + \Big[54\zeta(3)\alpha^3 N_c^3 - 45\alpha^3 N_c^3 - 36\zeta(3)\alpha^2 N_c^3 + 216\alpha^2 N_c^3 - 24\alpha^2 N_c^2 N_f \\
& \quad + 42\zeta(3)\alpha N_c^3 + 109\alpha N_c^3 + 48\zeta(3)\alpha N_c^2 N_f - 76\alpha N_c^2 N_f + 564\zeta(3) N_c^3 \\
& \quad - 5196 N_c^3 + 48\zeta(3) N_c^2 N_f + 1876 N_c^2 N_f - 160 N_c N_f^2 - 72 N_f \Big] \frac{a^3}{192}
\end{aligned}
$$

$$
\begin{aligned}
&+\Big[3843\zeta(3)\alpha^4N_c^5+2070\zeta(5)\alpha^4N_c^5-8052\alpha^4N_c^5+72\alpha^4N_c^4N_f\\
&\quad-3456\zeta(3)\alpha^4N_c^3+1890\zeta(5)\alpha^4N_c^3+8298\zeta(3)\alpha^3N_c^5-6390\zeta(5)\alpha^3N_c^5\\
&\quad-9411\alpha^3N_c^5-576\zeta(3)\alpha^3N_c^4N_f+192\alpha^3N_c^4N_f+10152\zeta(3)\alpha^3N_c^3\\
&\quad-18900\zeta(5)\alpha^3N_c^3-108\zeta(3)\alpha^2N_c^5-32790\zeta(5)\alpha^2N_c^5+73071\alpha^2N_c^5\\
&\quad-2448\zeta(3)\alpha^2N_c^4N_f+3360\zeta(5)\alpha^2N_c^4N_f-32388\alpha^2N_c^4N_f+2880\alpha^2N_c^3N_f^2\\
&\quad+3888\zeta(3)\alpha^2N_c^3-22680\zeta(5)\alpha^2N_c^3+648\alpha^2N_c^3+1296\alpha^2N_c^2N_f\\
&\quad-40458\zeta(3)\alpha N_c^5+26790\zeta(5)\alpha N_c^5+88801\alpha N_c^5+1152\zeta(3)\alpha N_c^4N_f\\
&\quad-12480\zeta(5)\alpha N_c^4N_f-42248\alpha N_c^4N_f+3072\zeta(3)\alpha N_c^3N_f^2+3136\alpha N_c^3N_f^2\\
&\quad-1080\zeta(3)\alpha N_c^3-52380\zeta(5)\alpha N_c^3+2592\alpha N_c^3-1728\zeta(3)\alpha N_c^2N_f\\
&\quad+3600\alpha N_c^2N_f+310041\zeta(3)N_c^5+192240\zeta(5)N_c^5-1888893N_c^5\\
&\quad-91728\zeta(3)N_c^4N_f-59040\zeta(5)N_c^4N_f+997068N_c^4N_f+13824\zeta(3)N_c^3N_f^2\\
&\quad-141984N_c^3N_f^2-526176\zeta(3)N_c^3+549990\zeta(5)N_c^3+14904N_c^3\\
&\quad+3840N_c^2N_f^3+54720\zeta(3)N_c^2N_f-34560\zeta(5)N_c^2N_f-104928N_c^2N_f\\
&\quad-4608\zeta(3)N_cN_f^2+18816N_cN_f^2-432N_f\Big]\frac{a^4}{4608N_c}\;+\;O(a^5)\,. \qquad (4.2)
\end{aligned}
$$

Finally, the quark anomalous dimension is

$$
\begin{aligned}
\gamma_\psi^{\mathrm{mMOM}}(a,\alpha) \;=\; & \alpha[N_c^2-1]\frac{a}{2N_c}\\
&+\Big[3\alpha^2N_c^4-3\alpha^2N_c^2+6\alpha N_c^4-6\alpha N_c^2+22N_c^4-4N_c^3N_f-19N_c^2\\
&\quad+4N_cN_f-3\Big]\frac{a^2}{8N_c^2}\\
&+\Big[18\zeta(3)\alpha^3N_c^6+15\alpha^3N_c^6-18\zeta(3)\alpha^3N_c^4-3\alpha^3N_c^4-12\alpha^3N_c^2\\
&\quad-90\zeta(3)\alpha^2N_c^6+105\alpha^2N_c^6+24\alpha^2N_c^5N_f+90\zeta(3)\alpha^2N_c^4\\
&\quad-87\alpha^2N_c^4-24\alpha^2N_c^3N_f-18\alpha^2N_c^2-618\zeta(3)\alpha N_c^6+431\alpha N_c^6\\
&\quad+96\zeta(3)\alpha N_c^5N_f-32\alpha N_c^5N_f+618\zeta(3)\alpha N_c^4-467\alpha N_c^4-96\zeta(3)\alpha N_c^3N_f\\
&\quad+32\alpha N_c^3N_f+36\alpha N_c^2-1182\zeta(3)N_c^6+3231N_c^6+192\zeta(3)N_c^5N_f\\
&\quad-944N_c^5N_f+32N_c^4N_f^2+894\zeta(3)N_c^4-2637N_c^4-192\zeta(3)N_c^3N_f\\
&\quad+968N_c^3N_f-32N_c^2N_f^2+288\zeta(3)N_c^2-576N_c^2-24N_cN_f-18\Big]\frac{a^3}{96N_c^3}\\
&+\Big[330\zeta(3)\alpha^4N_c^8+75\zeta(5)\alpha^4N_c^8-151\alpha^4N_c^8-12\alpha^4N_c^7N_f-420\zeta(3)\alpha^4N_c^6\\
&\quad-45\zeta(5)\alpha^4N_c^6+280\alpha^4N_c^6+12\alpha^4N_c^5N_f+42\zeta(3)\alpha^4N_c^4-30\zeta(5)\alpha^4N_c^4\\
&\quad-105\alpha^4N_c^4+48\zeta(3)\alpha^4N_c^2-24\alpha^4N_c^2+1116\zeta(3)\alpha^3N_c^8-180\zeta(5)\alpha^3N_c^8\\
&\quad-33\alpha^3N_c^8-192\zeta(3)\alpha^3N_c^7N_f+184\alpha^3N_c^7N_f-1620\zeta(3)\alpha^3N_c^6\\
&\quad+180\zeta(5)\alpha^3N_c^6+663\alpha^3N_c^6+192\zeta(3)\alpha^3N_c^5N_f-184\alpha^3N_c^5N_f\\
&\quad+360\zeta(3)\alpha^3N_c^4-558\alpha^3N_c^4+144\zeta(3)\alpha^3N_c^2-72\alpha^3N_c^2-1662\zeta(3)\alpha^2N_c^8\\
&\quad-192\zeta(3)N_c^3N_f+3130\zeta(5)\alpha^2N_c^8-5767\alpha^2N_c^8-384\zeta(3)\alpha^2N_c^7N_f\\
&\quad-160\zeta(5)\alpha^2N_c^7N_f+2580\alpha^2N_c^7N_f-96\alpha^2N_c^6N_f^2+1518\zeta(3)\alpha^2N_c^6\\
&\quad-4510\zeta(5)\alpha^2N_c^6+6307\alpha^2N_c^6+576\zeta(3)\alpha^2N_c^5N_f+160\zeta(5)\alpha^2N_c^5N_f\\
&\quad-2940\alpha^2N_c^5N_f+96\alpha^2N_c^4N_f^2+144\zeta(3)\alpha^2N_c^4+1380\zeta(5)\alpha^2N_c^4\\
&\quad-558\alpha^2N_c^4-192\zeta(3)\alpha^2N_c^3N_f+360\alpha^2N_c^3N_f+18\alpha^2N_c^2
\end{aligned}
$$

$$
\begin{aligned}
& - 33680\zeta(3)\alpha N_c^8 + 17420\zeta(5)\alpha N_c^8 + 14292\alpha N_c^8 + 7040\zeta(3)\alpha N_c^7 N_f \\
& - 3200\zeta(5)\alpha N_c^7 N_f - 3752\alpha N_c^7 N_f + 320\alpha N_c^6 N_f^2 + 49248\zeta(3)\alpha N_c^6 \\
& - 37500\zeta(5)\alpha N_c^6 - 13642\alpha N_c^6 - 9408\zeta(3)\alpha N_c^5 N_f + 5760\zeta(5)\alpha N_c^5 N_f \\
& + 2632\alpha N_c^5 N_f - 192\alpha N_c^4 N_f^2 - 15568\zeta(3)\alpha N_c^4 + 20080\zeta(5)\alpha N_c^4 \\
& - 122\alpha N_c^4 + 2368\zeta(3)\alpha N_c^3 N_f - 2560\zeta(5)\alpha N_c^3 N_f + 1024\alpha N_c^3 N_f \\
& - 128\alpha N_c^2 N_f^2 - 528\alpha N_c^2 + 96\alpha N_c N_f - 172560\zeta(3) N_c^8 + 61875\zeta(5) N_c^8 \\
& + 252104 N_c^8 + 43392\zeta(3) N_c^7 N_f - 12000\zeta(5) N_c^7 N_f - 118572 N_c^7 N_f \\
& - 1536\zeta(3) N_c^6 N_f^2 + 15392 N_c^6 N_f^2 + 118002\zeta(3) N_c^6 - 29085\zeta(5) N_c^6 \\
& - 225318 N_c^6 - 640 N_c^5 N_f^3 - 36864\zeta(3) N_c^5 N_f + 480\zeta(5) N_c^5 N_f \\
& + 120868 N_c^5 N_f + 2304\zeta(3) N_c^4 N_f^2 - 16592 N_c^4 N_f^2 + 66462\zeta(3) N_c^4 \\
& - 55830\zeta(5) N_c^4 - 19731 N_c^4 + 640 N_c^3 N_f^3 - 6912\zeta(3) N_c^3 N_f \\
& + 11520\zeta(5) N_c^3 N_f + 7152 N_c^3 N_f - 768\zeta(3) N_c^2 N_f^2 + 1200 N_c^2 N_f^2 \\
& - 2304\zeta(3) N_c^2 + 7680\zeta(5) N_c^2 - 3974 N_c^2 + 384\zeta(3) N_c N_f - 9448 N_c N_f \\
& - 9600\zeta(3) + 15360\zeta(5) - 3081 \Big] \frac{a^4}{384 N_c^4} ~+~ O(a^5) ~. \qquad (4.3)
\end{aligned}
$$

For practical purposes it is perhaps more appropriate to provide the explicit numerical values for all renormalization group functions at four loops for the $SU(3)$ colour group. Thus

$$
\begin{aligned}
\beta^{\mathrm{mMOM}}(a,\alpha) &= [0.666667 N_f - 11.000000] a^2 \\
&+ \Big[- 2.250000\alpha^3 - \alpha^2 N_f + 7.500000\alpha^2 - \alpha N_f + 9.750000\alpha \\
&\quad + 12.666667 N_f - 102.000000 \Big] a^3 \\
&+ \Big[0.375000\alpha^4 N_f - 15.468750\alpha^4 - 2.250000\alpha^3 N_f - 5.547924\alpha^3 \\
&\quad - 29.170372\alpha^2 N_f + 151.166003\alpha^2 - 35.750000\alpha N_f + 174.652162\alpha \\
&\quad - 19.383310 N_f^2 + 625.386670 N_f - 3040.482287 \Big] a^4 \\
&+ \Big[2.250000\alpha^5 N_f - 114.265701\alpha^5 + 4.816305\alpha^4 N_f - 298.616620\alpha^4 \\
&\quad - 61.925145\alpha^3 N_f - 37.364446\alpha^3 + 43.033474\alpha^2 N_f^2 - 1495.393156\alpha^2 N_f \\
&\quad + 5540.175086\alpha^2 + 3.385091\alpha N_f^3 - 83.916446\alpha N_f^2 - 152.826933\alpha N_f \\
&\quad - 1111.191853\alpha + 27.492640 N_f^3 - 1625.402243 N_f^2 + 24423.330550 N_f \\
&\quad - 100541.058601 \Big] a^5 ~+~ O(a^6) \\
\gamma_A^{\mathrm{mMOM}}(a,\alpha) &= [1.500000\alpha + 0.666667 N_f - 6.500000] a \\
&+ \Big[- 2.250000\alpha^3 - \alpha^2 N_f + 6.375000\alpha^2 - \alpha N_f + 6.375000\alpha \\
&\quad + 11.166667 N_f - 63.750000 \Big] a^2 \\
&+ \Big[0.375000\alpha^4 N_f - 15.468750\alpha^4 - 2.250000\alpha^3 N_f - 5.727087\alpha^3 \\
&\quad - 26.920372\alpha^2 N_f + 100.182677\alpha^2 - 22.784256\alpha N_f - 77.661754\alpha \\
&\quad - 14.383310 N_f^2 + 444.852414 N_f - 1769.783563 \Big] a^3 \\
&+ \Big[2.250000\alpha^5 N_f - 114.265701\alpha^5 + 3.972555\alpha^4 N_f - 270.114333\alpha^4 \\
&\quad - 72.936261\alpha^3 N_f + 287.225188\alpha^3 + 31.783474\alpha^2 N_f^2 \\
&\quad - 1126.940395\alpha^2 N_f + 3789.309068\alpha^2 + 3.385091\alpha N_f^3
\end{aligned}
$$

$$
\begin{aligned}
& - 124.724674\alpha N_f^2 + 1081.645997\alpha N_f - 6926.344667\alpha + 22.492640N_f^3 \\
& - 1141.450868N_f^2 + 14846.203053N_f - 54060.225189\Big] a^4 \; + \; O(a^5) \\
\gamma_c^{\mathrm{mMOM}}(a,\alpha) \;=\; & [0.750000\alpha - 2.250000]\, a \\
& + \Big[1.687500\alpha^2 - 1.687500\alpha + 0.750000N_f - 19.125000\Big] a^2 \\
& + \Big[2.799995\alpha^3 - 1.125000\alpha^2 N_f + 24.289587\alpha^2 - 0.857872\alpha N_f \\
& + 22.427774\alpha - 2.500000N_f^2 + 90.267128N_f - 635.349362\Big] a^3 \\
& + \Big[0.421875\alpha^4 N_f - 26.892596\alpha^4 - 2.931942\alpha^3 N_f - 121.006552\alpha^3 \\
& + 5.625000\alpha^2 N_f^2 - 185.757176\alpha^2 N_f + 648.958841\alpha^2 + 13.337341\alpha N_f^2 \\
& - 314.266897\alpha N_f + 1090.833574\alpha + 2.500000N_f^3 - 241.975687N_f^2 \\
& + 4788.563749N_f - 23240.416706]\, a^4 \; + \; O(a^5) \\
\gamma_\psi^{\mathrm{mMOM}}(a,\alpha) \;=\; & 1.333333\alpha a + \Big[3.000000\alpha^2 + 6.000000\alpha - 1.333333N_f + 22.333333\Big] a^2 \\
& + \Big[9.492589\alpha^3 + 2.000000\alpha^2 N_f - 0.296280\alpha^2 + 6.949789\alpha N_f \\
& - 78.967792\alpha + 0.888889N_f^2 - 59.211534N_f + 459.481285\Big] a^3 \\
& + \Big[- 0.750000\alpha^4 N_f + 61.650284\alpha^4 - 2.924683\alpha^3 N_f + 210.615979\alpha^3 \\
& - 2.00000\alpha^2 N_f^2 + 121.134099\alpha^2 N_f - 869.566016\alpha^2 + 6.962963\alpha N_f^2 \\
& + 77.834748\alpha N_f - 1553.524496\alpha - 4.444444N_f^3 + 281.560058N_f^2 \\
& - 4934.050066N_f + 20300.851595]\, a^4 \; + \; O(a^5) \\
\gamma_m^{\mathrm{mMOM}}(a,\alpha) \;=\; & - 4.0a + \Big[\alpha^2 + 1.333333N_f - 69.666667\Big] a^2 \\
& + \Big[1.916667\alpha^3 - 2.000000\alpha^2 N_f + 98.522232\alpha^2 + 2.000000\alpha N_f \\
& + 130.753633\alpha - 2.666667N_f^2 + 152.122739N_f - 1520.596003\Big] a^3 \\
& + \Big[0.750000\alpha^4 N_f - 36.419738\alpha^4 - 10.500000\alpha^3 N_f + 127.577470\alpha^3 \\
& + 6.000000\alpha^2 N_f^2 - 345.848075\alpha^2 N_f + 3588.203465\alpha^2 + 20.550022\alpha N_f^2 \\
& - 551.791827\alpha N_f + 5040.515124\alpha + 5.632797N_f^3 - 156.909331N_f^2 \\
& - 1073.781658N_f - 9337.969739]\, a^4 \; + \; O(a^5)
\end{aligned}
\tag{4.4}
$$

We have checked that the Landau gauge expression for the β-function agrees with that of [6].

One interesting consequence of these expressions is that we can provide the anomalous dimension of a particular dimension two operator which is

$$
\mathcal{O} \;=\; \tfrac{1}{2} A^a_\mu A^{a\,\mu} \;-\; \alpha \bar{c}^a c^a \; . \tag{4.5}
$$

It is known, [30, 31, 32], that $\mathcal{O}$ has a novel renormalization property. In the Landau gauge the anomalous dimension of $\mathcal{O}$ is the sum of the gluon and ghost anomalous dimensions. Moreover, in an arbitrary linear and nonlinear covariant gauge there is a simple generalization of this Slavnov-Taylor identity which was established in [33]. This was based on the observation given in [34] which were explicit three loop $\overline{\mathrm{MS}}$ computations. The operator is of interest as it was an attempt to have a gluon mass term in the Lagrangian which while not gauge invariant is in fact BRST invariant, [35]. It has seen renewed interest more recently, since it is believed to be the origin of dimension two power corrections in the running of an effective coupling constant in the low energy limit, [8, 9]. In [26] the four loop $\overline{\mathrm{MS}}$ Landau gauge result was given. However, the

arbitrary α $\overline{\mathrm{MS}}$ expression for a linear covariant gauge fixing was not recorded. As the gluon and ghost propagators are examined in the minimal MOM scheme, [36], it is worth providing the renormalization for the operators explicitly. Though for reasons of space we provide the $SU(3)$ expression†

$$
\begin{aligned}
\left.\gamma_{\mathcal{O}}^{\overline{\mathrm{MS}}}(a,\alpha)\right|_{SU(3)} &= [27\alpha + 8N_f - 105]\,\frac{a}{12} \\
&+ \left[108\alpha^2 + 567\alpha + 548N_f - 4041\right]\frac{a^2}{48} \\
&+ \left[12393\alpha^3 + 4374\zeta(3)\alpha^2 + 52488\alpha^2 - 22356\alpha N_f + 17496\zeta(3)\alpha \right. \\
&\quad + 268272\alpha - 12080N_f^2 - 28512\zeta(3)N_f + 437304N_f + 13122\zeta(3) \\
&\quad \left. - 2041389\right]\frac{a^3}{1728} \\
&+ \left[- 3011499\zeta(3)\alpha^4 + 4133430\zeta(5)\alpha^4 + 8030664\alpha^4 + 20824614\zeta(3)\alpha^3 \right. \\
&\quad + 708588\zeta(4)\alpha^3 - 10431990\zeta(5)\alpha^3 + 39936807\alpha^3 - 2939328\zeta(3)\alpha^2 N_f \\
&\quad + 314928\zeta(4)\alpha^2 N_f - 8669268\alpha^2 N_f + 93612348\zeta(3)\alpha^2 \\
&\quad - 3779136\zeta(4)\alpha^2 - 18305190\zeta(5)\alpha^2 + 159478227\alpha^2 \\
&\quad + 3359232\zeta(3)\alpha N_f^2 - 2796768\alpha N_f^2 - 60046272\zeta(3)\alpha N_f \\
&\quad - 11337408\zeta(4)\alpha N_f - 127188144\alpha N_f + 612180666\zeta(3)\alpha \\
&\quad - 43696260\zeta(4)\alpha - 513923130\zeta(5)\alpha + 1146415923\alpha + 497664\zeta(3)N_f^3 \\
&\quad - 846464N_f^3 - 19558656\zeta(3)N_f^2 - 6158592\zeta(4)N_f^2 - 107248896N_f^2 \\
&\quad - 289945440\zeta(3)N_f + 104451120\zeta(4)N_f + 313061760\zeta(5)N_f \\
&\quad + 2082893580N_f + 72636831\zeta(3) - 46766808\zeta(4) - 1908463680\zeta(5) \\
&\quad \left. - 7232776173\right]\frac{a^4}{373248} ~+~ O(a^5)
\end{aligned}
\tag{4.6}
$$

for non-zero α. Equipped with this then the equivalent minimal MOM scheme expression is

$$
\begin{aligned}
\left.\gamma_{\mathcal{O}}^{\mathrm{mMOM}}(a,\alpha)\right|_{SU(3)} &= [27\alpha + 8N_f - 105]\,\frac{a}{12} \\
&+ \left[-108\alpha^3 - 48\alpha^2 N_f + 387\alpha^2 - 48\alpha N_f + 225\alpha + 572N_f - 3978\right]\frac{a^2}{48} \\
&+ \left[648\alpha^4 N_f - 26730\alpha^4 - 3888\alpha^3 N_f + 21870\zeta(3)\alpha^3 - 31347\alpha^3 \right. \\
&\quad + 3888\zeta(3)\alpha^2 N_f - 53136\alpha^2 N_f - 102060\zeta(3)\alpha^2 + 337770\alpha^2 \\
&\quad - 3888\zeta(3)\alpha N_f - 36180\alpha N_f - 115182\zeta(3)\alpha + 43011\alpha \\
&\quad - 1536\zeta(3)N_f^2 - 27328N_f^2 - 29088\zeta(3)N_f + 959652N_f \\
&\quad \left. + 696924\zeta(3) - 4993812\right]\frac{a^3}{1728} \\
&+ \left[93312\alpha^5 N_f + 279936\zeta(3)\alpha^5 - 1224720\zeta(5)\alpha^5 - 3805380\alpha^5 \right. \\
&\quad - 77760\zeta(3)\alpha^4 N_f - 272160\zeta(5)\alpha^4 N_f + 557928\alpha^4 N_f \\
&\quad + 11078613\zeta(3)\alpha^4 + 1341360\zeta(5)\alpha^4 - 27025488\alpha^4 \\
&\quad + 1135296\zeta(3)\alpha^3 N_f + 77760\zeta(5)\alpha^3 N_f - 4591728\alpha^3 N_f \\
&\quad + 13097214\zeta(3)\alpha^3 + 1319490\zeta(5)\alpha^3 - 10218393\alpha^3 \\
&\quad - 221184\zeta(3)\alpha^2 N_f^2 + 1817280\alpha^2 N_f^2 + 14990832\zeta(3)\alpha^2 N_f
\end{aligned}
$$

†The full expression for $SU(N_c)$ is given in the attached data file.

$$
\begin{aligned}
& - 816480\zeta(5)\alpha^2 N_f - 71613396\alpha^2 N_f - 149908644\zeta(3)\alpha^2 \\
& + 83215350\zeta(5)\alpha^2 + 277974261\alpha^2 + 147456\zeta(3)\alpha N_f^3 \\
& - 36864\alpha N_f^3 - 6801408\zeta(3)\alpha N_f^2 + 3556224\alpha N_f^2 \\
& + 98413056\zeta(3)\alpha N_f - 10730880\zeta(5)\alpha N_f - 75346200\alpha N_f \\
& - 617646222\zeta(3)\alpha + 259312590\zeta(5)\alpha + 231547167\alpha \\
& - 73728\zeta(3)N_f^3 + 1125120N_f^3 + 8977920\zeta(3)N_f^2 \\
& + 14254080\zeta(5)N_f^2 - 82945888N_f^2 - 276910992\zeta(3)N_f \\
& - 280604160\zeta(5)N_f + 1438122060N_f + 1437422031\zeta(3) \\
& + 816720570\zeta(5) - 5780555523] \frac{a^4}{41472} ~+~ O(a^5)
\end{aligned}
\tag{4.7}
$$

for $SU(3)$.

5 Discussion.

We have provided all the renormalization group functions in QCD in the minimal momentum subtraction scheme introduced in [6]. To do this we have explicitly renormalized the theory and applied the renormalization prescription given in [6] to define the scheme. While [6] concentrated on the β-function the other renormalization group functions are required for other problems such as the infrared structure of propagators and therefore we have provided that information. Currently the results are known at four loops for the $SU(N_c)$ colour group and at three loops for a general group. One feature which differs from [6] rests in the renormalization of the gauge parameter. In [6] α was renormalized in the $\overline{\mbox{MS}}$ way whereas here we have chosen to follow a fuller approach and renormalize the gauge parameter according to the same criterion as all the 2-point functions. While this differs from [6] both sets of results agree in the Landau gauge which is the main gauge of interest for practical studies of the infrared dynamics of the gluon and ghost.

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References.

[1] G. 't Hooft, Nucl. Phys. **B61** (1973), 455.

[2] W.A. Bardeen, A.J. Buras, D.W. Duke & T. Muta, Phys. Rev. **D18** (1978), 3998.

[3] G. Martinelli, C. Pittori, C.T. Sachrajda, M. Testa & A. Vladikas, Nucl. Phys. **B445** (1995), 81.

[4] E. Franco & V. Lubicz, Nucl. Phys. **B531** (1998), 641

[5] W. Celmaster & R.J. Gonsalves, Phys. Rev. **D20** (1979), 1420.

[6] L. von Smekal, K. Maltman & A. Sternbeck, Phys. Lett. **B681** (2009), 336.

[7] J.C. Taylor, Nucl. Phys. **B33** (1971), 436.

[8] Ph. Boucaud, J.P. Leroy, J. Micheli, O. Pène & C. Roiesnel, JHEP **9810** (1998), 017.

[9] B. Blossier, Ph. Boucaud, M. Brinet, F. De Soto, V. Morenas, O. Pène, K. Petrov & J. Rodríguez-Quintero, arXiv:1301.7593.

[10] K.G. Chetyrkin & A. Rétey, hep-ph/0007088.

[11] J.A. Gracey, Nucl. Phys. **B662** (2003), 247.

[12] S.G. Gorishny, S.A. Larin, L.R. Surguladze & F.K. Tkachov, Comput. Phys. Commun. **55** (1989), 381.

[13] S.A. Larin, F.V. Tkachov & J.A.M. Vermaseren, "The Form version of Mincer", NIKHEF-H-91-18.

[14] J.A.M. Vermaseren, math-ph/0010025.

[15] P. Nogueira, J. Comput. Phys. **105** (1993), 279.

[16] D.J. Gross & F.J. Wilczek, Phys. Rev. Lett. **30** (1973), 1343; H.D. Politzer, Phys. Rev. Lett. **30** (1973), 1346.

[17] D.R.T. Jones, Nucl. Phys. **B75** (1974), 531.

[18] W.E. Caswell, Phys. Rev. Lett. **33** (1974), 244.

[19] O.V. Tarasov & A.A. Vladimirov, Sov. J. Nucl. Phys. **25** (1977), 585.

[20] E. Egorian & O.V. Tarasov, Theor. Math. Phys. **41** (1979), 863.

[21] S.A. Larin & J.A.M. Vermaseren, Phys. Lett. **B303** (1993), 334.

[22] O.V. Tarasov, A.A. Vladimirov & A.Yu. Zharkov, Phys. Lett. **B93** (1980), 429.

[23] T. van Ritbergen, J.A.M. Vermaseren & S.A. Larin, Phys. Lett. **B400** (1997), 379.

[24] K.G. Chetyrkin, Phys. Lett. **B404** (1997), 161.

[25] J.A.M. Vermaseren, S.A. Larin & T. van Ritbergen, Phys. Lett. **B405** (1997), 327.

[26] K.G. Chetyrkin, Nucl. Phys. **B710** (2005), 499.

[27] T. van Ritbergen, A.N. Schellekens & J.A.M. Vermaseren, Int. J. Mod. Phys. **A14** (1999), 41.

[28] O.V. Tarasov, JINR preprint P2-82-900.

[29] S.A. Larin, Phys. Lett. **B303** (1993), 113.

[30] R.M. Doria, F.A.B. Rabelo de Carvalho & S.P. Sorella, Braz. J. Phys. **20** (1990), 316.

[31] J.A. Gracey, Phys. Lett. **B552** (2003), 101.

[32] D. Dudal, H. Verschelde & S.P. Sorella, Phys. Lett. **B555** (2003), 126.

[33] N. Wschebor, Int. J. Mod. Phys. **A23** (2008), 2961.

[34] R.E. Browne, D. Dudal, J.A. Gracey, V.E.R. Lemes, M.S. Sarandy, R.F. Sobreiro, S.P. Sorella & H. Verschelde, J. Phys. **A39** (2006), 7889.

[35] G. Curci & R. Ferrari, Nuovo Cim. **A32** (1976), 151.

[36] A. Sternbeck & M. Müller-Preussker, arXiv:1211.3057.